

Thermal Transport in a Noncommutative Hydrodynamics

Michael Geracie and Dam Thanh Son

Kadanoff Center for Theoretical Physics, University of Chicago, Illinois 60637, Chicago, USA

We find the hydrodynamic equations of a system of particles constrained to be in the lowest Landau level. We interpret the hydrodynamic theory as a Hamiltonian system with the Poisson brackets between the hydrodynamic variables determined from the noncommutativity of space. We argue that the most general hydrodynamic theory can be obtained from this Hamiltonian system by allowing the Righi-Leduc coefficient to be an arbitrary function of thermodynamic variables. We compute the Righi-Leduc coefficient at high temperatures and show that it satisfies the requirements of particle-hole symmetry, which we outline.

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I. INTRODUCTION.

Interacting electrons in very high magnetic fields show extremely rich behaviors, the most well-known of which is the fractional quantum Hall (FQH) effect [1, 2]. In the most interesting limit, all the physics occurs in the lowest Landau level (LLL) and originates from the interactions.

In this paper, we study the finite-temperature dynamics of electrons in a magnetic field so high that all particles are constrained to be on the LLL. This problem is the finite-temperature counterpart of the FQH problem. While many quantum phenomena are smeared out by the temperature, the hydrodynamic theory, which takes hold at distances and time scales much larger than the mean free path/time, is expected to be universal. We assume the system is clean, without impurities, and the only relaxation mechanism is the interactions between particles. This regime is particularly relevant for the proposed realizations of the FQH regime in cold atomic gases [3–5] length/time. The main outcome of our investigation is the set of hydrodynamic equations [Eqs. (8), (18), and (30)] which describes the long-wavelength dynamics of the system, the identification of the kinetic coefficients, and the computation of the thermal Hall coefficient in the high-temperature regime [Eq. (34)].

Previous studies of transport in high magnetic field include Refs. [6–9]. In particular, in Ref. [9] a general approach based on conservation laws is developed for the hydrodynamics of a system in a quantizing magnetic field. This is the approach that we will follow in this paper. We concentrate here, however, on the LLL limit (the zero mass limit), which should be a regular limit when the particle carries a magnetic moment corresponding to the gyromagnetic factor $g = 2$. This allows us to consider the response of the system to variations of the magnetic field, as well as to discuss the particle-hole symmetry of the hydrodynamic equations.

An important concept in our discussion is that a particle in the lowest Landau level effectively lives on a noncommutative space [10], with its two coordinates x^1, x^2 satisfying the commutation law $[x^1, x^2] = -i\ell_B^2$. This idea has attracted some attention in the context of the quantum Hall effect; it has been speculated that the ap-

propriate field theory of the quantum Hall effect should be a noncommutative field theory (see, e.g. Refs. [11, 12]. For an introduction to noncommutative field theory, see Ref. [13].) We use this noncommutativity to argue for a particular Poisson bracket algebra between hydrodynamic variables, and proceed to derive the hydrodynamic equations from the Poisson brackets with the Hamiltonian. This approach is inspired by the Hamiltonian formulation of classical hydrodynamics [14]. The Hamiltonian equations that follow from the formalism form a self-consistent hydrodynamic theory, but we will argue that they need a slight modification to become the most general set of equations consistent with conservation laws and the second law of thermodynamics. This modification is related to the Righi-Leduc (thermal Hall) effect [15, 16].

II. THERMODYNAMICS AND CONSERVATION LAWS

Let us recall the basic thermodynamic functions of a system in an external magnetic field [9]. The grand potential is an extensive thermodynamic variable which depends on the temperature, chemical potential, and magnetic field: $\Omega = -VP(T, \mu, B)$. The partial derivatives of P are the entropy density, particle number density and magnetization: $dP = sdT + nd\mu + MdB$. The hydrodynamic pressure is not P but its Legendre transform with respect to B : $p = P - MB$, and hence $dp = sdT + nd\mu - BdM$. The energy density is $\varepsilon = Ts + \mu n - P$.

We consider a system of nonrelativistic particles of mass m and gyromagnetic factor $g = 2$ moving in a background magnetic field B , and will be interested in the regime where all higher Landau levels can be neglected. We will study the response of the system to arbitrary fluctuations of both electric and magnetic fields, assuming that B does not vanish at any place in space and time so that the separation between the lowest and the higher Landau levels is always maintained. The LLL limit corresponds to taking $m \rightarrow 0$ and all the physics should be finite in this limit for $g = 2$.

The Hamiltonian for our system is

$$H = \int d\mathbf{x} \left[\frac{|D_i\psi|^2}{2m} - \left(A_0 + \frac{B}{2m} \right) \psi^\dagger \psi \right] + \text{interactions}, \quad (1)$$

where $D_i = \partial_i - iA_i$ (we use units where $\hbar = c = 1$ and absorb the electron charge e into the gauge potential A_μ). We can also think about our system as that of particles with zero magnetic moment ($g = 0$), subjected to an external field in which the scalar potential is tuned to deviate from $B/2m$ by an amount which remains finite when $m \rightarrow 0$. The conservation laws are the conventional ones, with the replacement $A_0 \rightarrow A_0 + B/2m$,

$$\frac{\partial n}{\partial t} + \partial_i j^i = 0, \quad (2)$$

$$\frac{\partial}{\partial t}(m j^i) + \partial_k \Pi^{ik} = n \left(E_i + \frac{\partial_i B}{2m} \right) + \epsilon^{ik} j_k B, \quad (3)$$

$$\frac{\partial \varepsilon}{\partial t} + \partial_i \varepsilon^i = j^i \left(E_i + \frac{\partial_i B}{2m} \right). \quad (4)$$

We now extract the part divergent at $m \rightarrow 0$ from the conserved currents and the stress tensor in the following manner,

$$\tilde{j}^i = j^i + \frac{\epsilon^{ij}}{2m} \partial_j n, \quad (5)$$

$$\tilde{\Pi}^{ik} = \Pi_{ik} + \frac{1}{2} (\epsilon^{ij} \partial_j \tilde{j}^k + \epsilon^{kj} \partial_j \tilde{j}^i) - \frac{nB}{2m} \delta^{ik}, \quad (6)$$

$$\tilde{\varepsilon} = \varepsilon - \frac{nB}{2m}, \quad \tilde{\varepsilon}^i = \varepsilon^i - \frac{1}{2m} (B j^i + \epsilon^{ij} E_j n). \quad (7)$$

For the number current (5), this procedure of extracting the $1/m$ part was done in Ref. [19]. The conservation laws are regular in the $m \rightarrow 0$ limit in terms of the newly defined quantities,

$$\frac{\partial n}{\partial t} + \partial_i \tilde{j}^i = 0, \quad (8a)$$

$$\frac{\partial}{\partial t}(m \tilde{j}^i) + \partial_k \tilde{\Pi}^{ik} = n E_i + \epsilon^{ik} \tilde{j}_k B, \quad (8b)$$

$$\frac{\partial \tilde{\varepsilon}}{\partial t} + \partial_i \tilde{\varepsilon}^i = \tilde{j}^i E_i. \quad (8c)$$

These equations can also be obtained within the Newton-Cartan formalism [21]. Moreover, in the limit $m \rightarrow 0$ the first term in the left-hand side of Eq. (8b) can be dropped, and it becomes a force-balance condition. From now on we will drop the tildes in the finite currents. To close the equations we need to express j^i , ε^i and Π^{ik} through the derivatives of the local temperature and chemical potential. To first order in derivatives in the equations, we can limit ourselves to the leading order contribution to the stress tensor: $\Pi^{ik} = p \delta^{ik}$.

III. HAMILTONIAN MODEL OF A NONCOMMUTATIVE FLUID

We start with a simple model of particles moving in the lowest Landau level. We number the particles by

the index $A = 1 \dots N$, and the spatial coordinates by i . The coordinates of a particle do not commute with each other, but commute with those of other particles,

$$\{x_A^i, x_B^j\} = \delta_{AB} \frac{\epsilon^{ij}}{B(\mathbf{x}_A)}. \quad (9)$$

The particle number density,

$$n(\mathbf{x}) = \sum_A \delta(\mathbf{x} - \mathbf{x}_A), \quad (10)$$

then has the following Poisson bracket,

$$\{n(\mathbf{x}), n(\mathbf{y})\} = -\epsilon^{ij} \partial_i \left(\frac{n}{B} \right) \partial_j \delta(\mathbf{x} - \mathbf{y}). \quad (11)$$

We now need to understand the Poisson brackets involving the entropy density. Recall that in ideal hydrodynamics the entropy per particle s/n is conserved along fluid worldlines. We can assume that each particle A carries an entropy s_A for all time,

$$s(\mathbf{x}) = \sum_A s_A \delta(\mathbf{x} - \mathbf{x}_A). \quad (12)$$

as s' is in the continuum picture. We find

$$\{s(\mathbf{x}), n(\mathbf{y})\} = -\epsilon^{ij} \partial_i \left(\frac{s}{B} \right) \partial_j \delta(\mathbf{x} - \mathbf{y}), \quad (13)$$

$$\{s(\mathbf{x}), s(\mathbf{y})\} = -\epsilon^{ij} \partial_i c \partial_j \delta(\mathbf{x} - \mathbf{y}), \quad (14)$$

where

$$c = \sum_A \frac{s_A^2}{B(\mathbf{x}_A)} \delta(\mathbf{x} - \mathbf{x}_A). \quad (15)$$

In order to close the Poisson algebra, we should express c in terms of s and n . In the “mean field” approximation we may expect $c = s^2/nB$. We shall for now assume the most general c compatible with the Jacobi identity, which can be shown to be

$$c = \frac{n}{B} f\left(\frac{s}{n}\right). \quad (16)$$

Now the hydrodynamic equations can be obtained by computing Poisson brackets with the Hamiltonian

$$H = \int d\mathbf{x} [\varepsilon(s(\mathbf{x}), n(\mathbf{x}), B(\mathbf{x})) - A_0(\mathbf{x}) n(\mathbf{x})]. \quad (17)$$

Note that both the total particle number and the total entropy are Casimirs of the Poisson algebra, so they are automatically conserved. We find, for example, $\partial_t n = -\partial_i j^i$ where the particle number current j^i is

$$j^i = \frac{\epsilon^{ij}}{B} [n(E_j - \partial_j \mu) - s \partial_j T] + \epsilon^{ij} \partial_j \alpha. \quad (18)$$

where α cannot be determined from charge conservation alone. This can be done using the force balance equation (8b), into which we substitute $\Pi_{ik} = p \delta_{ik}$,

$$\partial_i p = n E_i + \epsilon^{ik} j_k B, \quad (19)$$

which, by using $dp = sdT + nd\mu - BdM$ completely determines j^i , and the result corresponds to $\alpha = M$. The first term on the right-hand side of Eq. (18) corresponds to the “transport current,” while the second part is the “magnetization current.”

Computing the Poisson bracket of s with the Hamiltonian, we can find the conservation law for the entropy,

$$\partial_t s + \partial_i s^i = 0, \quad s^i = \epsilon^{ij} \left[\frac{s}{B} (E_j - \partial_j \mu) - c \partial_j T \right]. \quad (20)$$

For energy density, we can use $\partial_t \varepsilon = T \partial_t s + \mu \partial_t n$ and derive from Eq. (8c) the energy current

$$\begin{aligned} \varepsilon^i = \epsilon^{ij} \left[\frac{\varepsilon + p}{B} (E_j - \partial_j \mu) - M \partial_j \mu - \left(\frac{\mu s}{B} + cT \right) \partial_j T + \partial_j M_E \right]. \end{aligned} \quad (21)$$

We have also introduced the “energy magnetization” M_E whose contribution to the energy current is divergence-free.

A. Středa formulas

We note here in passing that the Středa formula can be derived from our equation for the current. Expanding the current in terms of derivatives of thermodynamic variables, including the derivative of B ,

$$j^i = \epsilon^{ij} (\sigma_H E_j + \sigma_H^\mu \partial_j \mu + \sigma_H^T \partial_j T + \sigma_H^B \partial_j B), \quad (22)$$

we can then read out, for example

$$\begin{aligned} \sigma_H &= \frac{n}{B}, \\ \sigma_H^\mu &= -\frac{n}{B} + \left(\frac{\partial M}{\partial \mu} \right)_{TB} = -\frac{n}{B} + \left(\frac{\partial n}{\partial B} \right)_{\mu, T} \end{aligned} \quad (23)$$

where we have used a Maxwell’s relation. The naive Einstein relation $\sigma_H^\mu = -\sigma_H$ does not hold due to the non-vanishing magnetization current in thermal equilibrium. Note that in a zero-temperature incompressible phase n/B is constant and $\sigma_H^\mu = 0$, consistent with the expectation that small spatial variations of μ should not have any physical effect in such a phase. In thermal equilibrium the chemical potential traces the electric field, $\mu = A_0$, and so the only current flowing in the system is the magnetization current, equal to $j^i = \sigma_H^{\text{eq}} \epsilon^{ij} E_j$ where

$$\sigma_H^{\text{eq}} = \sigma_H + \sigma_H^\mu = \left(\frac{\partial n}{\partial B} \right)_{\mu, T}. \quad (24)$$

This is the Středa formula [20].

The thermopower can be read out from our expression for the transport current: it is equal to entropy per particle s/n , a known result [9]. Note also that a gradient of the magnetic field only leads to a magnetization (but not transport) current.

The noncommutative model above gives a complete expression for the energy current in terms of the function c appearing in the Poisson algebra and the energy magnetization M_E . Namely, if we write $\varepsilon^i = \epsilon^{ij} (\kappa_H E_j + \kappa_H^\mu \partial_j \mu + \kappa_H^T \partial_j T + \kappa_H^B \partial_j B)$, then

$$\kappa_H = \frac{\varepsilon + p}{B} \quad (25)$$

$$\kappa_H^\mu = -\frac{Ts + \mu n}{B} + \left(\frac{\partial M_E}{\partial \mu} \right)_{T, B}, \quad (26)$$

$$\kappa_H^T = -\frac{\mu s}{B} - cT + \left(\frac{\partial M_E}{\partial T} \right)_{\mu, B}, \quad (27)$$

$$\kappa_H^B = \left(\frac{\partial M_E}{\partial B} \right)_{T, \mu}. \quad (28)$$

IV. GENERALIZED THERMAL TRANSPORT

The Poisson bracket formalism, while giving a self-consistent set of equations, rely on certain unjustified assumptions. For example, the effect of dissipative heat conduction cannot be taken into account in this formalism. Fortunately, we can show that beside this effect, the most general hydrodynamic equations have the same forms as the equations derived above; the only modification is that there is now no restriction on the form of c , which to this point has been required to be of the form (16).

To write down the most general system of hydrodynamic equations, first we notice that the particle number current cannot be modified due to the force balance condition. Thus the only place where modifications can be made is in the constitutive relation for the energy current (21). The dissipative part has the familiar form of longitudinal heat conduction and shall not be discussed here. The most general additional transverse terms one can add to the energy current is $\epsilon^{ij} \Sigma_a \partial_j X^a$ where X^a , $a = 1, 2, 3$, are three independent thermodynamic variables (which can be chosen to be, e.g., μ , T , and B , but any other choice is equally valid) and $\Sigma_a = \Sigma_a(X)$ are the corresponding three kinetic coefficients. It is convenient to introduce the one-form $\Sigma \equiv \Sigma_a dX^a$ in the space of thermodynamic variables. The constraint on Σ_a is that one can modify the entropy current by adding to Eq. (20) a contribution of the form $\epsilon^{ij} \zeta_a \partial_j X^a$ and still preserve the entropy production rate, which should receive no contributions from these new kinetic terms. By direct calculation using the thermodynamic relation $ds = T^{-1}(d\varepsilon - \mu dn)$ and the conservation of energy and particle number, one can find the divergence of the new entropy current

$$\partial_t s + \partial_i s^i = \frac{1}{2} \epsilon^{ij} \left(d\zeta - \frac{1}{T} d\Sigma \right)_{ab} \partial_i X^a \partial_j X^b. \quad (29)$$

Therefore, for entropy conservation we need to have $d\zeta = T^{-1} d\Sigma$. The most general solution to this equation is $\zeta = db_0 + c_0 dT$, $\Sigma = d\sigma_0 + T c_0 dT$, where b_0 , c_0 and σ_0 are scalar functions (zero-forms) of thermodynamic

variables. In the energy current, σ_0 can be absorbed into the magnetization current, and c_0 into c to make the latter a unconstrained function of three thermodynamic variables. The full energy current therefore is

$$\varepsilon^i = \epsilon^{ij} \left[\frac{\varepsilon + p}{B} (E_j - \partial_j \mu) - M \partial_j \mu + \partial_j M_E - c_{\text{RL}} T \partial_j T \right], \quad (30)$$

where $c_{\text{RL}} = c + \mu s / BT$, corresponding to the Righi-Leduc effect with thermal Hall conductivity $K_H = T c_{\text{RL}}$. This means that in a gapped quantum Hall phase at low temperature $c_{\text{RL}} = \frac{\pi}{6} (c_R - c_L)$, where c_R and c_L are the numbers of right and left moving modes, respectively [15, 17, 18].

We emphasize here that the fact that the energy current is parametrized in terms of two functions c and M_E implies one relationship between the coefficient κ_H^μ , κ_H^T , and κ_H^B . The response to the Luttinger potential coupled to the energy density [22] can also be expressed in terms of these two functions [23].

Despite the fact that the hydrodynamic equations with generic c are dissipationless (without heat conduction), we are unable to find a Hamiltonian and a set of Poisson brackets that would lead to these equations. We leave the study of the Hamiltonian structure of our equations to future work.

A. Righi-Leduc coefficient at high temperature

At low temperature as μ changes c_{RL} is expected to vary in a complicated fashion as the system scans through many quantum Hall plateaux. When the temperature is large compared to the interaction energy, the system is weakly interacting and the Righi-Leduc coefficient c_{RL} can be computed reliably. One can follow the method of Ref. [6], but one can also employ the following short-cut. In thermal equilibrium, all states in the lowest Landau level has the same occupation number $\nu = (e^{-\beta\mu} + 1)^{-1}$, which depends only on μ/T but not μ and T separately. The current and energy current out of equilibrium, where μ and T vary in space, thus depend only on μ/T . But these quantities have nonzero dimension, and hence they have to vanish at high temperature. Thus at high temperatures and zero electric field, the terms on the right hand side of Eq. (30) cancel each other.

The grand partition function in this regime is

$$P = \frac{BT}{2\pi} \ln(1 + e^{\mu/T}), \quad (31)$$

from which all other thermodynamic potentials can be computed. In particular, $\varepsilon = p = 0$, $M = P/B$. The condition of vanishing energy current reads $M \partial_j \mu = \partial_j M_E - c_{\text{RL}} T \partial_j T$, which means

$$\frac{\partial M_E}{\partial \mu} = M, \quad \frac{\partial M_E}{\partial T} - T c_{\text{RL}} = 0. \quad (32)$$

The solution to these equations is

$$M_E = -\frac{1}{2\pi} T^2 \text{Li}_2(-e^{\mu/T}), \quad (33)$$

$$c_{\text{RL}} = -\frac{1}{2\pi} \left[\frac{\mu}{T} \ln(1 + e^{\mu/T}) + 2 \text{Li}_2(-e^{\mu/T}) \right] \\ = -\frac{1}{2\pi} \left[\ln \frac{\nu}{1-\nu} \ln \frac{1}{1-\nu} + 2 \text{Li}_2\left(-\frac{\nu}{1-\nu}\right) \right]. \quad (34)$$

The Righi-Leduc coefficient approaches 0 as $\nu \rightarrow 0$, and $\pi/6$ as $\nu \rightarrow 1$. The latter value matches exactly with that expected for the $\nu = 1$ integer quantum Hall state with a single chiral edge mode [15].

In the limit of low filling fraction $\nu \ll 1$ the formulas simplify. For example, $c_{\text{RL}} \approx -\frac{1}{2\pi} (2 - \mu/T) e^{\mu/T}$. It is interesting to note that, to leading order in $\ln(1/\nu)$,

$$c = c_{\text{RL}} - \frac{\mu s}{BT} \approx \frac{e^{\mu/T}}{2\pi} \frac{\mu^2}{T^2} = \frac{s^2}{nB}, \quad (35)$$

which is of the form (16) and moreover coincides with our initial “mean-field” guess for c .

B. Particle-hole symmetry

When the interaction between fermions is two-body, the system has particle-hole symmetry (for simplicity, here we assume the magnetic field is uniform and constant). This should also be a symmetry of the hydrodynamic equations. If one normalizes the one-body potential so that $\mu = 0$ corresponds to a half-filled Landau level ($\nu = 1/2$), then particle-hole symmetry is the symmetry under $\mu \rightarrow -\mu$. It is easy to check that the hydrodynamic equations are particle-hole symmetric if P , M_E and c_{RL} satisfy

$$P(T, \mu) = P(T, \mu) - \frac{B\mu}{2\pi}, \quad (36)$$

$$M_E(T, -\mu) = \frac{1}{4\pi} \left(\mu^2 + \frac{\mu^2}{3} T^2 \right) - M_E(T, \mu), \quad (37)$$

$$c_{\text{RL}}(T, -\mu) = \frac{\pi}{6} - c_{\text{RL}}(T, \mu). \quad (38)$$

In particular, at half filling $c_{\text{RL}} = \pi/12$ if particle-hole symmetry is not spontaneously broken, which should be the case at least at sufficiently high temperature. Note that these properties are satisfied by Eqs. (33) and (34).

V. CONCLUSION

We have shown that the full finite-temperature hydrodynamics of a system of particles confined to the lowest Landau level can be written down based on general principles. We have assumed that the interaction between the electrons are short-ranged. In the case of long-ranged Coulomb interaction, we have to add a Poisson equation

for the scalar potential, which can be done in a straightforward manner.

The Righi-Leduc coefficient c_{RL} should be viewed as a fundamental property of a system on the LLL at any filling fraction and temperature. In principle, the coefficient can be measured; but the task may be complicated by the edge transport, as well as from the contributions from the other terms in the energy current (30). We defer a more detailed study to future work.

To the order that we are working on, we are not sensitive to the first-order corrections to the stress tensor, including the dissipative shear and bulk viscosities and the dissipationless Hall viscosity. These can be introduced and it would be interesting to investigate their behaviors under particle-hole symmetry.

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